Bounds on the Elements of the Equivalent Network for Scattering in Waveguides

II. Application to Dielectric Obstacles

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FOR SCATTERING IN WAVEGUIDES
II. APPLICATION TO DIELECTRIC OBSTACLES

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Abstract

The theory developed in the companion article for obtaining rigorous bounds on \( \cot \eta \), where \( \eta \) is the phase shift, is applied to scattering by dielectric obstacles in rectangular waveguides. For the obstacles considered here, bounds are also obtained on the phase shifts directly and on the elements of the equivalent 'T' network. The exact solution for a dielectric slab of finite length which extends to the conducting boundaries of the waveguide and completely encloses the obstacle is introduced as a convenient trial function. The permittivity of the slab is retained as a parameter which is varied to improve the bounds. In the expressions for the bounds on \( \cot \eta_e \) and \( \cot \eta_o \), the particular obstacle configuration appears only in certain integrals of relatively simple form. Numerical results are obtained for large and small obstacles of various shapes, including some truly three-dimensional cases. The upper and lower bounds on the phase shifts and on the elements of the equivalent circuit are found to be quite close to one another. In one case, the bounds obtained by using the simple trial function are compared with the bounds obtained by using the trial function which generates the Schwinger integral variational principle.
# Table of Contents

1. **Introduction** | 1

2. **Trial function for dielectric obstacles** | 2

3. **Numerical examples for dielectric obstacles** | 7

References | 12

Distribution List
1. Introduction

In the preceding paper\(^1\) the Kato\(^2\) method for obtaining rigorous variational bounds on the phase shifts was adapted to scattering by obstacles in waveguides. In this paper the theory is applied to several specific examples of obstacles in a rectangular waveguide. The application to metallic obstacles is complicated by the requirement that the trial function \(u_t\) be normal to the surface of the obstacle, and it therefore seemed preferable to limit ourselves in the present paper to dielectric obstacles.

In principle, there is a great deal of freedom in the choice of a trial function. In practice, the trial function must of course be one for which the integrals in Equation (14), Part I, can be performed. We have gone further and sought a trial function which leads to an uncomplicated final analytic form from which numerical results can be fairly readily obtained. A particularly simple trial function which has been found to provide reasonably close bounds is the exact solution for a homogeneous dielectric slab of finite extent in the axial direction which completely encloses the obstacle and extends to the conducting boundaries of the waveguide. It should be noted that this trial function contains all of the symmetry properties that could be required with any obstacle to ensure completeness of the associated eigenfunctions.

We have also given some consideration to the trial function which generates the Schwinger integral variational principle.

Both the variational principle and the method for obtaining bounds are equally applicable to relatively large and very small obstacles, and include multiple, hollow and composite obstacles. For simplicity, we have considered only examples of uniform permittivity, but this is by no means necessary.
We have also considered obstacles whose maximum axial extent is a half wavelength in the dielectric: this enables us to use the lower bounds on the associated eigenvalues $\alpha_0$ and $\beta_0$ developed in Part I, but as noted there, bounds on $\alpha_0$ and on $\beta_0$ can be obtained under much more general conditions. Finally, the scattering problems satisfy the four conditions listed in the Introduction to Part I; these conditions are essential to the applicability of the method. The variational principle and the method for obtaining bounds are valid even when the problem cannot be reduced to a scalar problem, for example, when the obstacle does not extend between opposite walls of the waveguide.

2. **Trial function for dielectric obstacles**

In this section the theory of Part I will be applied to scattering from dielectric obstacles in a rectangular waveguide. The dimensions of the waveguide and the coordinate system are defined in Figure (1). The polarization of the dominant mode is parallel to the $y$ axis. We will take as the trial function the exact function for a homogeneous dielectric slab which fills the region $-d < z < d$, where $2d$ is the greatest extent of the obstacle in the $z$ direction, and extends to the conducting boundaries of the waveguide. Since the obstacle is symmetric about $z = 0$, the slab completely encloses the obstacle. The permittivity of the slab, or, equivalently, the 'potential' of the slab is retained as a parameter to be varied to improve the bounds on the phase shifts $\gamma_e$ and $\gamma_o$ associated with the even and odd standing wave solutions, respectively. The normalization constant $\Theta$ can also be varied. However, it is more convenient, and it will prove to be sufficient, simply to take $\Theta = \pi - kd$ in the odd case and in the determination of a lower bound
Figure 1

Rectangular waveguide showing dimensions and coordinate system.
on \( \text{kcot}(\eta - \varphi) \) in the even case, and \( \Theta = \frac{1}{2} \pi - kd \) in the determination of the upper bound in the even case. With this choice of \( \Theta \), \( \beta_\Theta \) can be regarded as infinite in Equation (27) of Part I. Note that the variation of either the potential or of \( \Theta \) to improve the upper bound is independent of the variation of these quantities to improve the lower bound.

The trial functions for \(-d < z < d\) with the above choices for \( \Theta \) are odd:

\[
\frac{U}{\rho_{ot}} = -\frac{1}{2} \left( \frac{2}{ab} \right)^{\frac{1}{2}} \frac{\sin(\pi x/a) \sin Kz}{\sin Kd} \tag{1a}
\]

even, lower bound:

\[
\frac{U}{\rho_{et}} = -\frac{1}{2} \left( \frac{2}{ab} \right)^{\frac{1}{2}} \frac{kd}{Kd} \frac{\sin(\pi x/a) \cos Kz}{\sin Kd} \tag{1b}
\]

even, upper bound:

\[
\frac{U}{\rho_{et}} = -\frac{1}{2} \left( \frac{2}{ab} \right)^{\frac{1}{2}} \frac{\sin(\pi x/a) \cos Kz}{\cos Kd} \tag{1c}
\]

where \( j \) is the unit vector in the y direction. The parameter \( K \) which is to be varied is related to the 'potential' \( W \) of the dielectric slab by

\[
K = (k^2 + W')^{\frac{1}{2}}. \tag{2}
\]

The normalization of the trial functions in Equations (1) is determined by matching the tangential components of \( E \) and \( H \) at \( z = \pm d \) to the asymptotic forms defined by Equations (9) and (13a) of Part I. The trial phase shifts,
\( \eta_{ot} \) and \( \eta_{et} \), are obtained by the same procedure. It is also convenient to take the weight function \( \rho \) to be constant for \(-d \leq z \leq d\) and to be zero for \(|z| > d\). Substitution of the trial functions and trial phase shifts in Equations (14) and (27) of Part I then yields as the bounds on \( \cot \eta_e \) and \( \cot \eta_o \),

\[
- \frac{1}{2} \csc^2(Kd) \left[ P^2 q^- + (2PR + R^2)(\frac{1}{2}) \right] / (\pi^2 - k^2 d^2 - R) \leq \\
Kd \cot(\eta_o - \pi + kd) - Kd \cot(Kd) + \frac{1}{2} \csc^2(Kd)(PQ^- + RI_0) \leq 0 \quad (3a)
\]

\[
Kd \cot(\eta_e - \frac{\pi}{2} + kd) \leq -Kd \tan(Kd) - \frac{1}{2} \sec^2(Kd)(PQ^+ + RI_e) \quad (3b)
\]

\[
KdKd \cot(Kd) - \frac{1}{2} Kd \csc^2(Kd)(PQ^+ + RI_e)
\]

\[
- \frac{1}{2} Kd \csc^2(Kd) \left[ P^2 q^+ + (2PR + R^2)(\frac{1}{2}) \right] / (\pi^2 - k^2 d^2 - R) \leq \\
(\frac{Kd}{\pi})^2 \cot(\eta_e - \pi + kd) \quad (3c)
\]

where

\[
P \equiv (kd)^2 - (Kd)^2 \quad (4a)
\]

\[
Q^+ \equiv 1 + \sin 2Kd / 2Kd \quad (4b)
\]

\[
R \equiv \omega d^2 \quad (4c)
\]
\[ I_\text{e,} = (2/\text{abd}) \int_{\text{obst}} \sin^2(\pi x/a) \sin^2(Kz) \, dt \quad (4d) \]

\[ I_\text{e} = (2/\text{abd}) \int_{\text{obst}} \sin^2(\pi x/a) \cos^2(Kz) \, dt \quad (4e) \]

The range of integration in Equations (4d) and (4e) is over the volume of the obstacle. With the specified values of \( \theta \), it follows from Equations (28) of Part I that a lower bound on \( K_\text{e} \) is given by

\[ K_\text{e} \geq \sin^2 - k^2 d^2 - Wi^2 \quad (5) \]

We have incorporated this lower bound in the inequalities (3).

The inequalities (3) are quite general; it can be seen that the specific geometry of the obstacle enters only in the integrals \( I_\text{e,} \) and \( I_\text{e} \) defined by Equations (4d) and (4e), respectively. The integrals \( I_\text{e,} \) and \( I_\text{e} \) are evaluated below for four obstacle geometries: a rectangular parallelepiped and right circular cylinders oriented parallel to each of the coordinate axes.

(A) A rectangular parallelepiped, extending from \( x_1 \) to \( x_2 \),

\( y_1 \) to \( y_2 \), and \( z = -d \) to \( z = +d \), as shown in Figure (A).

\[ I_{\text{e,}} = \left( y_2 - y_1 \right) \left[ \frac{x_2 - x_1}{a} - \frac{1}{2\pi} \sin \left( \frac{2\pi x_1}{a} \right) + \frac{1}{2\pi} \sin \left( \frac{2\pi x_2}{a} \right) \right] \]

\[ K_{\text{e,}} \geq \sin^2 - k^2 d^2 - Wi^2 \quad (5A) \]
A dielectric obstacle in a rectangular waveguide in the form of a rectangular parallelepiped oriented parallel to the axes of the waveguide.
(B) A right circular cylinder parallel to the x axis, extending from \( x_1 \) to \( x_2 \) and with radius \( d \), as shown in Figure (2B). The integrals do not depend upon the y coordinate of the axis of the cylinder.

\[
I_{e.o} = \frac{\pi d}{2b} \left[ \frac{x_2 - x_1}{a} - \frac{1}{2\pi} \sin \left( \frac{2\pi x_2}{a} \right) + \frac{1}{2\pi} \sin \left( \frac{2\pi x_1}{a} \right) \right] \\
\times \left[ 1 + \frac{J_1(2Kd)}{Kd} \right]
\]

(6B)

where \( J_1 \) is a Bessel function.

(C) A right circular cylinder parallel to the y axis, extending from \( y_1 \) to \( y_2 \), with radius \( d \), and with the axis of the cylinder at \( x = x_0 \), as shown in Figure (2C).

\[
I_{e.o} = \frac{\pi d(y_2 - y_1)}{2ab} \left\{ 1 + \frac{J_1(2Kd)}{Kd} - \cos \left( \frac{2\pi y_0}{a} \right) \cdot \frac{J_1(2\pi d/a)}{\pi d/a} \right\} \\
\times \\
\frac{J_1 \left( \frac{2[Kd]^2 + (\pi d/c)^2}{\left( [Kd]^2 + (\pi d/c)^2 \right)^{\frac{1}{2}}} \right)}{\left( [Kd]^2 + (\pi d/c)^2 \right)^{\frac{1}{2}}}
\]

(6C)

(D) A right circular cylinder parallel to the z axis, extending from \( z = -d \) to \( z = +d \) with radius \( R \) and with axis at \( x_o \).
Figure 2B

A dielectric obstacle in a rectangular waveguide.
In the form of a right circular cylinder of radius \( d \) parallel to the \( x \)-axis.
Figure 2C

A dielectric obstacle in a rectangular waveguide, in the form of a right circular cylinder of radius $d$ parallel to the $y$-axis.
as shown in Figure (2D). The integrals are independent of the y coordinate of the axis of the cylinder.

\[ I_{e.o} = \frac{\pi R^2}{ab} q^+ \left[ \left( \frac{2\pi x_0}{a} \right) \frac{J_1(2\pi R/a)}{\pi R/a} \right] . \] (6D)

On the basis of the above results, we can readily evaluate the necessary integrals for the case of hollow, composite and multiple obstacles composed of simple shapes like those described above. Terms of the form \( f(R)I \) are simply replaced by \( \sum_n f(R_n)I_n \) where the index \( n \) refers to a constituent obstacle. The set of obstacles must still be symmetrical about \( z = 0 \) and must not extend in the \( z \) direction more than the shortest half wavelength in the dielectric.

3. **Numerical examples for dielectric obstacles**

In this section, numerical examples are presented for each of the obstacle geometries discussed in the preceding section: a rectangular parallelepiped, and right circular cylinders parallel to the \( x, y \) and \( z \) axes, which will be referred to hereafter as geometries A, B, C and D, respectively. The following parameters have been chosen arbitrarily for convenience.

\[ \omega^2 \frac{a^2}{c^2} = 2\pi^2 \] (7a)

\[ b/a = 1/2 \] (7b)

\[ d/a = 1/4 \] (7c)

\[ \epsilon = 2\epsilon_0 \] (7d)
A dielectric obstacle in a rectangular waveguide, in the form of a right circular cylinder of radius $R$ parallel to the $z$-axis.
where \( \varepsilon_0 \) is the permittivity of free space. Note that with this choice of the parameters only the dominant mode propagates, since

\[
k_n = \left[ \frac{\omega^2}{c^2} - \frac{n^2 \kappa^2}{a^2} \right]^{\frac{1}{2}}
\]

is real only for \( n = 1 \). Also, a half guide wavelength in the dielectric is \( a/\sqrt{\varepsilon} \) which exceeds \( 2d = \frac{1}{2} a \) by the factor \( 2/\sqrt{\varepsilon} \). In addition, the following dimensions have been selected for the particular geometries:

**Geometry (A):** \( x_1 = \frac{1}{2} a - d, \ x_2 = \frac{1}{2} a + d, \ y_1 = 0 \ y_2 = b \)

**Geometry (B):** \( x_1 = 0, \ x_2 = a \).

**Geometry (C):** \( x_0 = \frac{1}{2} a, \ y_1 = 0, \ y_2 = b \).

**Geometry (D):** \( x_0 = \frac{1}{2} a \). Two cases are calculated, (1) \( R = \frac{1}{4} a \) and (2) \( R = \frac{1}{8} a \).

The parameter \( Kd/\pi \) might be expected to yield the closest bounds for some value between \( 1/4 \) and \( 3\frac{1}{2}/4 \), corresponding to no obstacle and to a dielectric slab with the same permittivity as the obstacle, respectively. Calculations were carried out for \( Kd/\pi = 0.25, 0.30, 0.35, 0.40 \) and \( 0.45 \). The best of the upper and lower bounds on \( \eta_0 \) and \( \eta_e \) for each geometry are listed in Table I. The corresponding values for \( Kd/\pi \) are listed in Table II. Bounds on \( \cot(\eta - \Theta) \) are actually calculated, so that in general bounds on \( \eta \) are determined only to within an integral multiple of \( \pi \). However, in all of the examples considered we have required that the obstacle be such that \( -\pi < \eta - \Theta < 0 \) in order to obtain lower bounds on \( \alpha_0 \) and \( \beta_0 \). In addition, it follows from the monotonicity theorem that \( 0 < \eta \) since the permittivity is positive in all examples considered. As a consequence, bounds can be
Table I.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>$\eta_0$ lower bound</th>
<th>$\eta_0$ upper bound</th>
<th>$\eta_e$ lower bound</th>
<th>$\eta_e$ upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$18^{0}42'$</td>
<td>$19^{0}24'$</td>
<td>$34^{0}20'$</td>
<td>$34^{0}55'$</td>
</tr>
<tr>
<td>B</td>
<td>$13^{0}0'$</td>
<td>$13^{0}47'$</td>
<td>$35^{0}49'$</td>
<td>$36^{0}46'$</td>
</tr>
<tr>
<td>C</td>
<td>$11^{0}19'$</td>
<td>$12^{0}12'$</td>
<td>$33^{0}13'$</td>
<td>$33^{0}50'$</td>
</tr>
<tr>
<td>C*</td>
<td>$11^{0}25'$</td>
<td>-</td>
<td>$32^{0}19'$</td>
<td>-</td>
</tr>
<tr>
<td>D1</td>
<td>$14^{0}30'$</td>
<td>$15^{0}30'$</td>
<td>$31^{0}8'$</td>
<td>$32^{0}18'$</td>
</tr>
<tr>
<td>D2</td>
<td>$3^{0}19'$</td>
<td>$3^{0}47'$</td>
<td>$12^{0}6'$</td>
<td>$13^{0}25'$</td>
</tr>
</tbody>
</table>

Upper and lower bounds on the even and odd phase shifts were calculated with the simple trial function with the exception of $C^*$, which was calculated for geometry C using the trial function which generates the Schwinger integral variational principle.
Table II.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Kd/π (odd case)</th>
<th>Kd/π (even case)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lower bound</td>
<td>upper bound</td>
</tr>
<tr>
<td>A</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>B</td>
<td>0.40</td>
<td>0.30</td>
</tr>
<tr>
<td>C</td>
<td>0.40</td>
<td>0.30</td>
</tr>
<tr>
<td>D1</td>
<td>0.40</td>
<td>0.35</td>
</tr>
<tr>
<td>D2</td>
<td>0.30</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Bounds on the phase shifts calculated with the simple trial function using five values of the variation parameter Kd/π: 0.25, 0.30, 0.35, 0.40 and 0.45. The best of the five bounds in each case is listed in Table I; in this table are listed the corresponding values of Kd/π (note that these are not bounds on Kd/π).
determined on \( \eta \) directly as well as on \( \cot(\eta - \Theta) \). It is more conventional
to present the data in terms of the elements of the equivalent circuit
shown in Figure (3)\(^3,4\). The reactances \( X_1 \) and \( X_2 \) are relative to the char-
acteristic impedance of the waveguide, and are related to the phase shifts
\( \eta_0 \) and \( \eta_e \) by

\[
X_1 = \tan \eta_0 \quad (11a)
\]

\[
X_2 = \frac{1}{2}(\cot \eta_e + \tan \eta_0) \quad (11b)
\]

The upper and lower bounds on \( X_1 \) and \( X_2 \) for each of the geometries are
listed in Table III.

Geometries A, C and D1 correspond to obstacles of comparable volume,
whereas the volume of D2 is one quarter that of D1, resulting in a smaller
series reactance and larger shunt reactance. Obstacle B has twice the
volume of C and D1, but the additional material is in a region of low
field and makes relatively little contribution.

The upper and lower bounds on the even and odd phase shifts for
gometry D1, corresponding to all five values of the variation parameter
\( K_d/\pi \), are listed in Table IV. The best bounds were obtained with
\( K_d/\pi = 0.40 \) except for the upper bound on \( \eta_0 \), which was obtained with
\( K_d/\pi = 0.35 \). The best bounds with geometry D2 were all obtained with
\( K_d/\pi = 0.30 \). The same range of values of \( K_d \) was employed in calculating
both geometries (D1) and (D2), since then only the integrals \( I_e \) and \( I_o \)
of Equation (6D), which are independent of \( K_d \), were different for the two
cases. However, the above result suggests that better bounds might have
Figure 3

Equivalent 'T' network for describing the far field effects of scattering by dielectric obstacles in waveguides.
Table III.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>$X_1$ lower bound</th>
<th>$X_1$ upper bound</th>
<th>$X_2$ lower bound</th>
<th>$X_2$ upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.3348</td>
<td>0.3522</td>
<td>0.8833</td>
<td>0.9082</td>
</tr>
<tr>
<td>B</td>
<td>0.2309</td>
<td>0.2453</td>
<td>0.7846</td>
<td>0.8155</td>
</tr>
<tr>
<td>C</td>
<td>0.2001</td>
<td>0.2162</td>
<td>0.8460</td>
<td>0.8717</td>
</tr>
<tr>
<td>D1</td>
<td>0.2586</td>
<td>0.2773</td>
<td>0.9202</td>
<td>0.9664</td>
</tr>
<tr>
<td>D2</td>
<td>0.05795</td>
<td>0.06613</td>
<td>2.125</td>
<td>2.365</td>
</tr>
</tbody>
</table>

Upper and lower bounds on the reactances of the equivalent T network shown in Figure (3), for the geometries described in the text.
Table IV.

<table>
<thead>
<tr>
<th>Kd/λ</th>
<th>n₀ lower bound</th>
<th>n₀ upper bound</th>
<th>nₑ lower bound</th>
<th>nₑ upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>13°27'</td>
<td>15°58'</td>
<td>24°56'</td>
<td>90°9'</td>
</tr>
<tr>
<td>0.30</td>
<td>14°0'</td>
<td>15°41'</td>
<td>27°42'</td>
<td>48°6'</td>
</tr>
<tr>
<td>0.35</td>
<td>14°16'</td>
<td>15°30'</td>
<td>30°9'</td>
<td>35°37'</td>
</tr>
<tr>
<td>0.40</td>
<td>14°30'</td>
<td>16°31'</td>
<td>31°8'</td>
<td>32°18'</td>
</tr>
<tr>
<td>0.45</td>
<td>15°54'</td>
<td>18°14'</td>
<td>0°</td>
<td>34°16'</td>
</tr>
</tbody>
</table>

The upper and lower bounds on the even and odd phase shifts for geometry D1 are listed for all five values of the variation parameter Kd/λ.
been obtained for geometry (D2) if a smaller range of values of Kd close to \(\pi/4\) had been selected.

If the 'potential' of the slab is taken as the average of the actual 'potential' contained within the slab, weighted with the square of the trial field, it is evident that the first order correction term vanishes in Equation (14) of Part I. As a consequence, the trial phase shift is itself a variational approximation to the actual phase shift, and when \(\beta_0 = \infty\) it is a lower bound. It seems likely that the 'potential' which provides the best bounds is not very different from the average 'potential'. A rough estimate of the average 'potential' might provide a criterion for selecting the range of values of Kd in a particular problem. It would also follow that Kd should be close to \(\pi/4\) for small obstacles.

Approximate values of \(\eta_o\) and \(\eta_e\) were calculated for the example of geometry (3) using the Schwinger integral variational principle and trial function described in Reference (5) and are listed in Table I. If we take \(\rho = W\) in Equation (27) of Part I, and if we have \(W > 0\) and \(\eta < \pi\), then \(\beta_0 = 1\) as shown in Part I. We can combine Equations (14) and (27) of Part I to obtain an upper bound on \(\cot \eta\). It can be shown that the Schwinger variational principle provides a lower bound on \(\cot \eta\) by substituting in this inequality the trial function

\[
\frac{1}{(ab)^{1/2}} \left( 2\cot \eta \psi(r) + a \int G(r, r') W(r') \phi(r') \, dr' \right)
\]  

(12)

where \(\psi(r)\) is the field that would exist in the absence of the obstacle.
\( \phi(r) \) is the trial function used in the Schwinger variational principle, and \( G(r,r') \) is the appropriate Green's function defined in Reference (5). (This result was previously proved by Kato\(^2\)). The Schwinger result provides a better lower bound on \( \eta_o \) and a worse lower bound on \( \eta_e \) than the simple trial function; however, the comparison between the two trial functions is not altogether fair because the trial function for the Schwinger case was chosen to provide accurate results only for small obstacles.

We have seen that relatively close bounds can be obtained with a rather crude trial function and only five values of the variational parameter. Closer bounds might be obtained with the same trial function by varying \( \Theta \), or by considering other values of \( K_d \); the most natural way to choose another value of \( K_d \) would probably be to plot the bound on \( \cot(\eta-\Theta) \) versus \( K_d \) for the values of \( K_d \) originally considered, to guess from the curve the best value of \( K_d \), and then to do the calculation for that value of \( K_d \). It is of course also possible to introduce a more sophisticated trial function with more variational parameters, but this procedure will probably result in considerably more labor.
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2. T. Kato, Progress of Theoretical Physics, 6, 394 (1951).


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